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Semilocal E -preinvexity and its applications in nonlinear multiple objective fractional programming

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Abstract

Introduction: In this paper, a new class of functions called semilocal E -preinvex functions is introduced, which is generalization of semi- E -preinvex functions and semilocal E -convex functions. Some of its basic properties are obtained.

Methods and Materials: Using E - η -semidifferentiability, some optimality conditions and duality results are established for a nonlinear multiobjective fractional programming with semilocal E -preinvex and related functions.

Conclusion: The results presented in this paper extend and generalize previously known results in this area.

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1. Introduction

Convexity and generalized convexity play a vital role in the study of optimality and duality aspects of mathematical programming. Attempts have been made to generalize convexity to study their role in solving such types of problems. Generalizations of convexity related to optimality and duality for nonlinear singleobjective or multiobjective optimization problems have been of much interest in the recent past, and many contributions have been made to this development. See, e.g., [1-6] and the references therein.

After Ewing [7] presented the definition of semilocal convexity, by using more general semilocal preinvexity and η -semidifferentiability, Preda and Stancu-Minasian [8] gave optimality conditions for weak vector minima and extended the Wolfe and Mond-Weir duals, generalizing results of Preda and Stancu-Minasian [9]. Based on the results of [8,9], Preda further established optimality conditions and duality results for a nonlinear fractional multiple objective programming problem with semilocal preinvex functions involving η -semidifferentiability in [10].

On the other hand, Youness [11] proposed the concepts of E -convex sets, E -convex functions, and E -convex programming, discussed some of their basic properties, and obtained some optimality results on E -convex programming. Subsequently, Chen [12] brought forward a class of semi- E -convex functions and also discussed its basic properties. Moreover, by combining the concepts of semi- E -convexity and semilocal

convexity, Hu et al. [13] put forward the concept of semilocal E -convexity, studied its some characterizations, and established some optimality conditions and duality results for semilocal E -convex programming. In [14], optimality and duality were further studied for a fractional multiple objective programming involving semilocal E -convexity. Recently, Fulga and Preda [15] extended the E -convexity to E -preinvexity and local E -preinvexity and studied some of their properties and an application. More recently, Luo and Jian [16] presented semi- E -preinvex maps in Banach spaces and discussed their properties.

Motivated by research works of [13-16] and references therein, in this paper, we introduce the concept of semilocal E -preinvexity and study some of its important properties. Then, we establish optimality conditions for a nonlinear fractional multiple objective programming problem (VFP) involving weaker E - η -semidifferentiable functions. Finally, a general Bector type dual for (VFP) is formulated, and duality results are obtained using semilocal E -preinvex and related functions. The concept of semilocal E -preinvexity unifies the concepts of semilocal E -convexity and semi- E -preinvexity. Thus, we extend the work of [10,13,14] and generalize results obtained in the literatures on this topic.

2. Notations and definitions

Throughout the paper, the convention below will be followed.

Let R^n denote the n -dimensional Euclidean space, $E: R^n \rightarrow R^n$ and $\eta: R^n \times R^n \rightarrow R^n$ be two fixed mappings.

For any $x, y \in R^n$, by $x \leq y$, we mean $x_i \leq y_i$ for all i , by $x < y$, we mean $x_i < y_i$ for all i and $x_j < y_j$ for at least one $j \in \{1, 2, \dots, n\}$, by $x <_y$, we mean $x_i < y_i$ for all i , and by $x \not\leq y$, we mean the negation of $x \leq y$.

In this section, we review some related definitions that will be used in the sequel.

Definition 2.1. [15] A set $K \subset R^n$ is said to be E -invex with respect to η if

$$E(y) + \lambda \eta(E(x), E(y)) \in K, \quad \forall x, y \in K, \lambda \in [0, 1].$$

Definition 2.2. [16] Let $K \subset R^n$ be an E -invex set with respect to η . A function $f: R^n \rightarrow R$ is said to be semi- E -preinvex on K with respect to η if

$$f(E(y) + \lambda \eta(E(x), E(y))) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \forall x, y \in K, \lambda \in [0, 1].$$

Definition 2.3. [13] A set $K \subset R^n$ is said to be local starshaped E -convex, if there is a map E such that corresponding to each pair of points $x, y \in K$, there is a maximal positive number $a(x, y) \leq 1$ satisfying

$$\lambda E(x) + (1 - \lambda)E(y) \in K, \quad \forall \lambda \in [0, a(x, y)]. \quad (2.1)$$

Definition 2.4. [13] A function $f: R^n \rightarrow R$ is said to be semilocal E -convex on a local starshaped E -convex set $K \subset R^n$ if for each pair of $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (2.1)), there exists a positive number $b(x, y) \leq a(x, y)$ satisfying

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \forall \lambda \in [0, b(x, y)].$$

Definition 2.5. [15] A set $K \subset R^n$ is said to be local E -invex with respect to η , if $\forall x, y \in K$, there exists $a(x, y) \in (0, 1]$ such that, $\forall \lambda \in [0, a(x, y)]$,

$$E(y) + \lambda\eta(E(x), E(y)) \in K. \quad (2.2)$$

Remark 2.1. Every local starshaped E -convex set is a local E -invex set with respect to η , where $\eta(x, y) = x - y$, $\forall x, y \in R^n$. Every E -invex set with respect to η is a local E -invex set with respect to η , where $a(x, y) = 1$, $\forall x, y \in R^n$. But their converses are not necessarily true.

Example 2.1. Let $K = [-4, -1) \cup [1, 4]$,

$$E(x) = \begin{cases} x^2, & \text{if } |x| \leq 2, \\ -1, & \text{if } |x| > 2. \end{cases} \quad (2.3)$$

$$\eta(x, y) = \begin{cases} x - y, & \text{if } x \geq 0, y \geq 0, \text{ or } x \leq 0, y \leq 0, \\ -1 - y, & \text{if } x > 0, y \leq 0, \text{ or } x \leq 0, y < 0, \\ 1 - y, & \text{if } x < 0, y \geq 0, \text{ or } x \leq 0, y > 0. \end{cases} \quad (2.4)$$

We can testify that K is a local E -invex set with respect to η . However, when $x_0 = 1$, $y_0 = 3$, there is a $\lambda_1 \in [0, 1]$ such that $E(y_0) + \lambda_1\eta(E(x_0), E(y_0)) = -1 \notin K$, i.e., K is not an E -invex set with respect to η .

Similarly, for any positive number $a \leq 1$, there exists a sufficiently small positive number $\lambda_2 \leq a$ satisfying $\lambda_2 E(x_0) + (1 - \lambda_2)E(y_0) = -1 + 2\lambda_2 \notin K$, that is, K is not a local starshaped E -convex set.

Definition 2.6. [15] A function $f: R^n \rightarrow R$ is said to be local E -preinvex on $k \subset R^n$ with respect to η if for any $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (2.2)), there exists $0 < b(x, y) \leq a(x, y)$ such that K is a local E -invex set and

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y)), \quad \forall \lambda \in [0, b(x, y)].$$

Definition 2.7. [17] A point $\bar{x} \in X$ is said to be an efficient solution for programming problem (MP): $\min F(x) = (F_1(x), \dots, F_p(x))$, $x \in X$, if there exists no $x \in X$ such that $F_i(x) < F_i(\bar{x})$ and $F_i(x) \leq F_i(\bar{x})$ for some $i \in \{1, 2, \dots, p\}$. A point $\bar{x} \in X$ is said to be a weak efficient solution for (MP), if there exists no $x \in X$ such that $F(x) < F(\bar{x})$.

3. Semilocal E -preinvexity

In the section, we introduce the concepts of semilocal E -preinvex and related functions and study some of their properties.

Definition 3.1. A function $f: R^n \rightarrow R$ is said to be semilocal E -preinvex on $k \subset R^n$ with respect to η if for any $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (2.2)), there exists $0 < b(x, y) \leq a(x, y)$ such that K is a local E -invex set and

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \forall \lambda \in [0, b(x, y)].$$

If

$$f(E(y) + \lambda\eta(E(x), E(y))) \geq \lambda f(x) + (1 - \lambda)f(y), \quad \forall \lambda \in [0, b(x, y)],$$

then f is called as a semilocal E -preincave function on k .

If the inequality signs above are strict for any $x, y \in K$ and $x \neq y$, then f is called as a strict semilocal E -preinvex (E -preincave) function.

Remark 3.1. Every semilocal E -convex function is a semilocal E -preinvex function, where $\eta(x, y) = x - y$, $\forall x, y \in R^n$. Every semi- E -preinvex function with respect to η is a

semilocal E -preinvex function, where $a(x, y) = b(x, y) = 1, \forall x, y \in R^n$. But their converses are not necessarily true.

We give below an example of semilocal E -preinvex function, which is neither a semilocal E -convex function nor a semi- E -preinvex function.

Example 3.1. Let the map $E: R \rightarrow R$ be defined as

$$E(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } 1 < x \leq 2, \\ x, & \text{if } 0 \leq x \leq 1 \text{ or } x > 2, \end{cases} \quad (3.1)$$

and the map $\eta: R \times R \rightarrow R$ be defined as

$$\eta(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1 - x, & \text{if } x \neq y. \end{cases} \quad (3.2)$$

Obviously, R is a local starshaped E -convex set and a local E -invex set with respect to η . Let $f: R \rightarrow R$ be defined as

$$f(x) = \begin{cases} 0, & \text{if } 1 < x \leq 2, \\ 1, & \text{if } x > 2, \\ -x + 1, & \text{if } 0 \leq x \leq 1, \\ -x + 2, & \text{if } x < 0. \end{cases} \quad (3.3)$$

We can prove that f is semilocal E -preinvex on R with respect to η . However, when $x_0 = 2, y_0 = 3$, and for any $b \in (0, 1]$, there exists a sufficiently small $\lambda_0 \in (0, b]$ satisfying

$$f(\lambda_0 E(x_0) + (1 - \lambda_0)E(y_0)) = f(3 - 2\lambda_0) = 1 > 1 - \lambda_0 = \lambda_0 f(x_0) + (1 - \lambda_0)f(y_0).$$

That is, $f(x)$ is not a semilocal E -convex function on R .

Similarly, taking $x_1 = 1, y_1 = 4$, we have

$$f(E(y_1) + \lambda_1 \eta(E(x_1), E(y_1))) = f(4) = 1 > 1 - \lambda_1 = \lambda_1 f(x_1) + (1 - \lambda_1)f(y_1),$$

for some $\lambda_1 \in [0, 1]$.

Thus, $f(x)$ is not a semi- E -preinvex function on R with respect to η .

Definition 3.2. A real-valued function f defined on a local E -invex set $K \subset R^n$ is said to be quasi-semilocal E -preinvex (with respect to η) if for all $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (2.2)) satisfying $f(x) \leq f(y)$, there is a positive number $b(x, y) \leq a(x, y)$ such that

$$f(E(y) + \lambda \eta(E(x), E(y))) \leq f(y), \quad \forall \lambda \in [0, b(x, y)].$$

Definition 3.3. A real-valued function f defined on a local E -invex set $K \subset R^n$ is said to be pseudo-semilocal E -preinvex (with respect to η) if for all $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (2.2)) satisfying $f(x) < f(y)$, there are a positive number $b(x, y) \leq a(x, y)$ and a positive number $c(x, y)$ such that

$$f(E(y) + \lambda \eta(E(x), E(y))) \leq f(y) - \lambda c(x, y), \quad \forall \lambda \in [0, b(x, y)].$$

Remark 3.2. Every semilocal E -preinvex function on a local E -invex set K with respect to η is both a quasi-semilocal E -preinvex function and a pseudo-semilocal E -preinvex function.

Definition 3.4. Let $f: K \rightarrow R$ be a function, where $K \subset R^n$ is a local E -invex set with respect to η . We say that f is E - η -semidifferentiable at $\bar{x} \in K$ if $E(\bar{x}) = \bar{x}$ and

$$f^+(\bar{x}; \eta(E(x), \bar{x})) = \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} [f(\bar{x} + \lambda \eta(E(x), \bar{x})) - f(\bar{x})], \text{ exists for each } x \in K, \text{ where}$$

$$f^+(\bar{x}; \eta(E(x), \bar{x})) = \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} [f(\bar{x} + \lambda \eta(E(x), \bar{x})) - f(\bar{x})],$$

(the right derivative at \bar{x} along the direction $\eta(E(x), \bar{x})$).

Remark 3.3. If E is an identity map, the E - η -semidifferentiability is the η -semidifferentiability notion [18]. If E is an identity map and $\eta(x, \bar{x}) = x - \bar{x}$, the E - η -semidifferentiability is the semidifferentiability notion. If a function is directionally differentiable, then it is semidifferentiable, but the converse is not true.

Using Definition 3.1-3.4, the following lemma is immediate.

Lemma 3.1. (i). Let f be semilocal E -preinvex (E -preincave) and E - η -semidifferentiable at $\bar{x} \in K \subset R^n$, where K is a local E -invex set with respect to η . Then,

$$f(x) - f(\bar{x}) \geq (\leq) f^+(\bar{x}; \eta(E(x), \bar{x})), \quad \forall x \in K.$$

(ii). Let f be quasi(pseudo)-semilocal E -preinvex and E - η -semidifferentiable at $\bar{x} \in K \subset R^n$, where K is a local E -invex set with respect to η . Then,

$$f(x) \leq (<) f(\bar{x}) \Rightarrow f^+(\bar{x}; \eta(E(x), \bar{x})) \leq (<) 0, \quad \forall x \in K.$$

Theorem 3.1. Let $f: K \subset R^n \rightarrow R$ be a local E -preinvex function on a local E -invex set K with respect to η , then f is a semilocal E -preinvex function if and only if $f(E(x)) \leq f(x)$, $\forall x \in K$.

Proof. Suppose that f is a semilocal E -preinvex function on set K with respect to η , then for each pair of points $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (2.2)), there exists a positive number $b(x, y) \leq a(x, y)$ satisfying

$$f(E(x) + \lambda \eta(E(y), E(x))) \leq \lambda f(y) + (1 - \lambda)f(x), \quad \lambda \in [0, b(x, y)].$$

By letting $\lambda = 0$, we have $f(E(x)) \leq f(x)$, $\forall x \in K$.

Conversely, assume that f is a local E -preinvex function on a local E -invex set K , then for any $x, y \in K$, there exist $a(x, y) \in (0, 1]$ satisfying (2.2) and $b(x, y) \in (0, a(x, y)]$ such that

$$f(E(y) + \lambda \eta(E(x), E(y))) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y)), \quad \forall \lambda \in [0, b(x, y)].$$

Since $f(E(x)) \leq f(x)$, $\forall x \in K$, then

$$f(E(y) + \lambda \eta(E(x), E(y))) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \forall \lambda \in [0, b(x, y)].$$

The proof is completed.

Remark 3.4. A local E -preinvex function on a local E -invex set with respect to η is not necessarily a semilocal E -preinvex function.

Example 3.2. Let $K = [-4, -1) \cup [1, 4]$,

$$E(x) = \begin{cases} x^2, & \text{if } |x| \leq 2, \\ -1, & \text{if } |x| > 2, \end{cases} \quad (3.4)$$

$$\eta(x, y) = \begin{cases} x - y, & \text{if } x \geq 0, y \geq 0, \text{ or } x \leq 0, y \leq 0, \\ -1 - y, & \text{if } x > 0, y \leq 0, \text{ or } x \leq 0, y < 0, \\ 1 - y, & \text{if } x < 0, y \geq 0, \text{ or } x \leq 0, y > 0, \end{cases} \quad (3.5)$$

and $f: R \rightarrow R$ be defined by $f(x) = x^2$, then f is local E -preinvex on K with respect to η .

Since $f(E(2)) = 16 > f(2) = 4$, from Theorem 3.1, it follows that f is not a semilocal E -preinvex function.

Definition 3.5. The set $G = \{(x, \alpha): x \in K \subset R^n, \alpha \in R\}$ is said to be a local E -invex set with respect to η corresponding to R^n if there are two maps η, E and a maximal positive number $a((x, \alpha_1), (y, \alpha_2)) \leq 1$, for each $(x, \alpha_1), (y, \alpha_2) \in G$ such that

$$(E(y) + \lambda\eta(E(x), E(y)), \lambda\alpha_1 + (1 - \lambda)\alpha_2) \in G, \quad \forall \lambda \in [0, a((x, \alpha_1), (y, \alpha_2))].$$

Theorem 3.2. Let $K \subset R^n$ be a local E -invex set with respect to η . Then f is a semilocal E -preinvex function on K with respect to η if and only if its epigraph $G_f = \{(x, \alpha): x \in K, f(x) \leq \alpha, \alpha \in R\}$ is a local E -invex set with respect to η corresponding to R^n .

Proof. Assume that f is semilocal E -preinvex on K with respect to η and $(x, \alpha_1), (y, \alpha_2) \in G_f$, then $x, y \in K, f(x) \leq \alpha_1, f(y) \leq \alpha_2$. Since K is a local E -invex set, there is a maximal positive number $a(x, y) \leq 1$ such that

$$E(y) + \lambda\eta(E(x), E(y)) \in K, \quad \forall \lambda \in [0, a(x, y)].$$

In addition, in view of f being a semilocal E -preinvex function on K with respect to η , there is a positive number $b(x, y) \leq a(x, y)$ such that

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda f(x) + (1 - \lambda)f(y) \leq \lambda\alpha_1 + (1 - \lambda)\alpha_2, \quad \forall \lambda \in [0, b(x, y)].$$

That is, $(E(y) + \lambda\eta(E(x), E(y)), \lambda\alpha_1 + (1 - \lambda)\alpha_2) \in G_f, \forall \lambda \in [0, b(x, y)]$.

Therefore, $G_f = \{(x, \alpha): x \in K, f(x) \leq \alpha, \alpha \in R\}$ is a local E -invex set with respect to η corresponding to R^n .

Conversely, if G_f is a local E -invex set with respect to η corresponding to R^n , then for any points $(x, f(x)), (y, f(y)) \in G_f$, there is a maximal positive number $a((x, f(x)), (y, f(y))) \leq 1$ such that

$$(E(y) + \lambda\eta(E(x), E(y)), \lambda f(x) + (1 - \lambda)f(y)) \in G_f, \quad \forall \lambda \in [0, a((x, f(x)), (y, f(y)))].$$

That is, $E(y) + \lambda\eta(E(x), E(y)) \in K$,

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \forall \lambda \in [0, a((x, f(x)), (y, f(y)))].$$

Thus, K is a local E -invex set and f is a semilocal E -preinvex function on K .

Theorem 3.3. If f is a semilocal E -preinvex function on a local E -invex set $K \subset R^n$ with respect to η , then the level set $S_\alpha = \{x \in K: f(x) \leq \alpha\}$ is a local E -invex set for any $\alpha \in R$.

Proof. For any $\alpha \in R$ and $x, y \in S_\alpha$, then $x, y \in K$ and $f(x) \leq \alpha, f(y) \leq \alpha$. Since K is a local E -invex set, there is a maximal positive number $a(x, y) \leq 1$ such that

$$E(y) + \lambda\eta(E(x), E(y)) \in K, \quad \forall \lambda \in [0, a(x, y)].$$

In addition, due to the semilocal E -preinvexity of f , there is a positive number $b(x, y) \leq a(x, y)$ such that

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda f(x) + (1 - \lambda)f(y) \leq \lambda\alpha + (1 - \lambda)\alpha = \alpha, \quad \forall \lambda \in [0, b(x, y)].$$

That is, $E(y) + \lambda\eta(E(x), E(y)) \in S_\alpha, \forall \lambda \in [0, b(x, y)]$.

Therefore, S_α is a local E -invex set with respect to η for any $\alpha \in R$.

Theorem 3.4. *Let f be a real-valued function defined on a local E -invex set $K \subset R^n$. Then f is a semilocal E -preinvex function with respect to η if and only if for each pair of points $x, y \in K$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (2.2)), there exists a positive number $b(x, y) \leq a(x, y)$ such that*

$$f(E(y) + \lambda\eta(E(x), E(y))) < \lambda\alpha + (1 - \lambda)\beta, \quad \forall \lambda \in [0, b(x, y)],$$

whenever $f(x) < \alpha, f(y) < \beta$.

Proof. Let $x, y \in K$ and $\alpha, \beta \in R$ such that $f(x) < \alpha, f(y) < \beta$. Due to the local E -invexity of K , there is a maximal positive number $a(x, y) \leq 1$ such that

$$E(y) + \lambda\eta(E(x), E(y)) \in K, \quad \forall \lambda \in [0, a(x, y)].$$

In addition, owing to the semilocal E -preinvexity of f , there is a positive number $b(x, y) \leq a(x, y)$ such that

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda f(x) + (1 - \lambda)f(y) < \lambda\alpha + (1 - \lambda)\beta, \quad \forall \lambda \in [0, b(x, y)].$$

Conversely, let $(x, \alpha) \in G_f, (y, \beta) \in G_f$ (see epigraph G_f in Theorem 3.2), then $x, y \in K, f(x) \leq \alpha, f(y) \leq \beta$. Hence, $f(x) < \alpha + \epsilon$ and $f(y) < \beta + \epsilon$ hold for any $\epsilon > 0$. According to the hypothesis, for $x, y \in K$ (with a positive number $a(x, y) \leq 1$ satisfying (2.2)), there exists a positive number $b(x, y) \leq a(x, y)$ such that

$$f(E(y) + \lambda\eta(E(x), E(y))) < \lambda\alpha + (1 - \lambda)\beta + \epsilon, \quad \forall \lambda \in [0, b(x, y)].$$

Let $\epsilon \rightarrow 0^+$, then

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda\alpha + (1 - \lambda)\beta, \quad \forall \lambda \in [0, b(x, y)].$$

That is, $(E(y) + \lambda\eta(E(x), E(y)), \lambda\alpha + (1 - \lambda)\beta) \in G_f, \forall \lambda \in [0, b(x, y)]$. Therefore, G_f is a local E -invex set corresponding to R^n .

From Theorem 3.2, it follows that f is semilocal E -preinvex on K with respect to η .

4. Optimality criteria

In this section, we establish sufficient optimality conditions for fractional multiobjective programming with semilocal E -preinvex and related functions under E - η -semidifferentiability assumptions.

We consider the following nonlinear fractional multiobjective programming problem:

$$(VFP) \begin{cases} \text{minimize } \frac{f(x)}{g(x)} = \left(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_p(x)}{g_p(x)} \right) \\ \text{subject to } h_i(x) \leq 0, \quad j \in M = \{1, 2, \dots, m\}, \\ x \in X_0, \end{cases} \quad (4.1)$$

where $X_0 \subseteq R^n$ is a local E -invex set and $g_i(x) > 0$ for all $x \in X_0$ and each $i \in P = \{1, 2, \dots, p\}$. Let $f = (f_1, f_2, \dots, f_p)$, $g = (g_1, g_2, \dots, g_p)$ and $h = (h_1, h_2, \dots, h_m)$, and denote $X = \{x: h_j(x) \leq 0, j \in M, x \in X_0\}$, the feasible set of problem (VFP). For $x^* \in X$, we put $M(x^*) = \{j: h_j(x^*) = 0, j \in M\}$, $N(x^*) = M \setminus M(x^*)$.

Utilizing the parametric approach of Jagannathan [19], we also formulate the nonlinear multi-objective programming problem as follows:

$$(VFP_u) \begin{cases} \text{minimize } (f_1(x) - u_1 g_1(x), \dots, f_p(x) - u_p g_p(x)) \\ \text{subject to } h_j(x) \leq 0, \quad j \in M, \\ x \in X_0, \end{cases} \quad (4.2)$$

where $u = (u_1, u_2, \dots, u_p) \in \mathbb{R}^p$.

The following lemma connects the weak efficient solutions for (VFP) and (VFP)_u.

Lemma 4.1. *A point x^* is a weak efficient solution for (VFP) if and only if x^* is a weak efficient solution for (VFP)_{u*}, where $u^* = (u_1^*, \dots, u_p^*) = \left(\frac{f_1(x^*)}{g_1(x^*)}, \dots, \frac{f_p(x^*)}{g_p(x^*)} \right)$.*

Proof. Suppose that there exists a feasible point $x \in X$, such that for each $i \in P$,

$$\begin{aligned} f_i(x) - u_i^* g_i(x) &< f_i(x^*) - u_i^* g_i(x^*) \\ \Rightarrow f_i(x) - u_i^* g_i(x) &< f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x^*) = 0 \\ \Rightarrow f_i(x) &< \frac{f_i(x^*)}{g_i(x^*)} g_i(x) \Rightarrow \frac{f_i(x)}{g_i(x)} < \frac{f_i(x^*)}{g_i(x^*)}, \end{aligned}$$

which contradicts the weak efficiency of x^* for (VFP).

On the other hand, if there exists a feasible point $x \in X$ such that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(x^*)}{g_i(x^*)} = u_i^*,$$

then

$$f_i(x) - u_i^* g_i(x) < 0 = f_i(x^*) - u_i^* g_i(x^*), \quad \forall i \in P,$$

and this is a contradiction to the weak efficiency of x^* for (VFP)_{u*}.

Thus, the proof is completed.

Now, we establish some sufficient optimality conditions for the problem (VFP).

Theorem 4.1. *Let $\bar{x} \in X$, $E(\bar{x}) = \bar{x}$, and f, h be semilocal E -preinvex and g be semilocal E -preincave, and they are all E - η -semidifferentiable at \bar{x} . Further, suppose that there are $\lambda^0 = (\lambda_1^0, \lambda_2^0, \dots, \lambda_p^0) \in \mathbb{R}^p$, and $v^0 = (v_1^0, v_2^0, \dots, v_m^0) \in \mathbb{R}^m$ such that*

$$\lambda^{0T} f^+(\bar{x}; \eta(E(x), \bar{x})) + v^{0T} h^+(\bar{x}; \eta(E(x), \bar{x})) \geq 0, \quad \forall x \in X, \quad (4.3)$$

$$g_i^+(\bar{x}; \eta(E(x), \bar{x})) \leq 0, \quad \forall x \in X, \quad i \in P, \quad (4.4)$$

$$v^{0T} h(\bar{x}) = 0, \quad (4.5)$$

$$\lambda^0 \geq 0, \quad v^0 \geq 0. \quad (4.6)$$

Then \bar{x} is a weak efficient solution for (VFP).

Proof. By contradiction, suppose that \bar{x} is not a weak efficient solution for (VFP), then there exists a point $\hat{x} \in X$ such that

$$\frac{f_i(\hat{x})}{g_i(\hat{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})}, \quad i \in P. \quad (4.7)$$

By the above hypotheses and Lemma 3.1, we have

$$f_i(\hat{x}) - f_i(\bar{x}) \geq f_i^+(\bar{x}; \eta(E(\hat{x}), \bar{x})), \quad i \in P, \quad (4.8)$$

$$g_i(\hat{x}) - g_i(\bar{x}) \leq g_i^+(\bar{x}; \eta(E(\hat{x}), \bar{x})), \quad i \in P, \quad (4.9)$$

$$h_j(\hat{x}) - h_j(\bar{x}) \geq h_j^+(\bar{x}; \eta(E(\hat{x}), \bar{x})), \quad j \in M. \quad (4.10)$$

Multiplying (4.8) by λ_i^0 , (4.10) by v_j^0 , and summing the obtained relations, we get

$$\sum_{i=1}^p \lambda_i^0 (f_i(\hat{x}) - f_i(\bar{x})) + \sum_{j=1}^m v_j^0 (h_j(\hat{x}) - h_j(\bar{x})) \geq \lambda^{0T} f^*(\bar{x}; \eta(E(\hat{x}), \bar{x})) + v^{0T} h^*(\bar{x}; \eta(E(\hat{x}), \bar{x})) \geq 0. \quad (4.11)$$

Since $\hat{x} \in X$, $v^0 \geq 0$, by (4.5) and (4.11), we further have

$$\sum_{i=1}^p \lambda_i^0 (f_i(\hat{x}) - f_i(\bar{x})) \geq 0. \quad (4.12)$$

Utilizing (4.6) and (4.12), we know that there exists at least an i_0 ($1 \leq i_0 \leq p$) such that

$$f_{i_0}(\hat{x}) \geq f_{i_0}(\bar{x}). \quad (4.13)$$

On the other hand, (4.4) and (4.9) imply

$$g_i(\hat{x}) \leq g_i(\bar{x}), \quad i \in P. \quad (4.14)$$

Now, using (4.13), (4.14), and $g > 0$, we obtain

$$\frac{f_{i_0}(\hat{x})}{g_{i_0}(\hat{x})} \geq \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})}, \quad (4.15)$$

which is in contradiction with (4.7).

Hence, the theorem is proved.

Similar to the proof of Theorem 4.1, we have the following theorem.

Theorem 4.2. Let $\bar{x} \in X$, $E(\bar{x}) = \bar{x}$, and f, g, h be E - η -semidifferentiable at \bar{x} . If there exist $\lambda^0 \in R^p$ and $v^0 \in R^m$ such that the conditions (4.3)-(4.6) hold and $\lambda^{0T} f(x) + v^{0T} h(x)$ is a semilocal E -preinvex function, then \bar{x} is a weak efficient solution for (VFP).

Theorem 4.3. Suppose that $\bar{x} \in X$, $E(\bar{x}) = \bar{x}$, and $u_i^0 = \frac{f_i(\bar{x})}{g_i(\bar{x})}$ ($i \in P$). Further suppose that $(f_i(x) - u_i^0 g_i(x))$ ($i \in P$) are all pseudo-semilocal E -preinvex functions and $h_j(x)$ ($j \in M(\bar{x})$) are all quasi-semilocal E -preinvex functions and f, g, h are all E - η -semi-differentiable at \bar{x} . If there exist $\lambda^0 \in R^p$ and $v^0 \in R^m$ such that

$$\sum_{i=1}^p \lambda_i^0 (f_i^*(\bar{x}; \eta(E(x), \bar{x})) - u_i^0 g_i^*(\bar{x}; \eta(E(x), \bar{x}))) + v^{0T} h^*(\bar{x}; \eta(E(x), \bar{x})) \geq 0, \quad \forall x \in X, \quad (4.16)$$

$$v^{0T} h(\bar{x}) = 0, \quad (4.17)$$

$$\lambda^0 \geq 0, \quad v^0 \geq 0, \quad (4.18)$$

then \bar{x} is a weak efficient solution for (VFP).

Proof. We assume, by contradiction, that \bar{x} is not a weak efficient solution for (VFP). Then, there exists $\tilde{x} \in X$ such that

$$\frac{f_i(\tilde{x})}{g_i(\tilde{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})}, \quad i.e., \quad f_i(\tilde{x}) - u_i^0 g_i(\tilde{x}) < 0, \quad i \in P,$$

which means

$$f_i(\tilde{x}) - u_i^0 g_i(\tilde{x}) < f_i(\bar{x}) - u_i^0 g_i(\bar{x}) < 0, \quad i \in P.$$

By the pseudo-semilocal E -preinvexity of $(f_i(x) - u_i^0 g_i(x))(i \in P)$ and Lemma 3.1, we have

$$f_i^+(\bar{x}; \eta(E(\bar{x}), \bar{x})) - u_i^0 g_i^+(\bar{x}; \eta(E(\bar{x}), \bar{x})) < 0, \quad i \in P.$$

Utilizing $\lambda^0 \geq 0$, we get

$$\sum_{i=1}^p \lambda_i^0 (f_i^+(\bar{x}; \eta(E(\bar{x}), \bar{x})) - u_i^0 g_i^+(\bar{x}; \eta(E(\bar{x}), \bar{x}))) < 0. \quad (4.19)$$

From $h(\bar{x}) \leq 0$ and $h_j(\bar{x}) = 0, j \in M(\bar{x})$, we have $h_j(\bar{x}) \leq h_j(\bar{x}), \forall j \in M(\bar{x})$. By the quasi-semilocal E -preinvexity of h_j and Lemma 3.1, we obtain

$$h_j^+(\bar{x}; \eta(E(\bar{x}), \bar{x})) \leq 0, \quad \forall j \in M(\bar{x}).$$

Considering $v^0 \geq 0$ and $v_j^0 = 0$ for $j \in N(\bar{x})$, one has

$$\sum_{j=1}^m v_j^0 h_j^+(\bar{x}; \eta(E(\bar{x}), \bar{x})) \leq 0. \quad (4.20)$$

Hence, by (4.19) and (4.20), we have

$$\sum_{i=1}^p \lambda_i^0 (f_i^+(\bar{x}; \eta(E(\bar{x}), \bar{x})) - u_i^0 g_i^+(\bar{x}; \eta(E(\bar{x}), \bar{x}))) + \sum_{j=1}^m v_j^0 h_j^+(\bar{x}; \eta(E(\bar{x}), \bar{x})) < 0,$$

which is in contradiction with relation (4.16) at point $\bar{x} \in X$.

Therefore, \bar{x} is a weak efficient solution for (VFP).

Analogously, we have the theorem below.

Theorem 4.4. Let $\bar{x} \in X, E(\bar{x}) = \bar{x}, u_i^0 = \frac{f_i(\bar{x})}{g_i(\bar{x})} (i \in P)$, and f, g, h be E - η -semidifferentiable at \bar{x} . If there exist $\lambda^0 \in R^p$ and $v^0 \in R^m$ such that the conditions (4.16)-(4.18) hold and $\sum_{i=1}^p \lambda_i^0 (f_i(x) - u_i^0 g_i(x)) + v_{M(\bar{x})}^{0T} h_{M(\bar{x})}(x)$ is a pseudo-semilocal E -preinvex function, then \bar{x} is a weak efficient solution for (VFP).

The following corollary is followed directly by Theorem 4.3 or Theorem 4.4.

Corollary 4.1. Let $\bar{x} \in X, E(\bar{x}) = \bar{x}$, and $u_i^0 = \frac{f_i(\bar{x})}{g_i(\bar{x})} (i \in P)$. Further suppose that $f, h_{M(\bar{x})}$ are all semilocal E -preinvex functions, g is a semilocal E -preincave function, and f, g, h are all E - η -semidifferentiable at \bar{x} . If there exist $\lambda^0 \in R^p$ and $v^0 \in R^m$ such that the conditions (4.16)-(4.18) hold, then \bar{x} is a weak efficient solution for (VFP).

5. Duality

Utilizing the approaches of Bector et al. [20], we formulate the dual problem for (VFP) as follows:

$$(VFD) \begin{cases} \text{maximize} & (\lambda_1, \lambda_2, \dots, \lambda_p) \\ \text{subject to} & \sum_{i=1}^p \alpha_i (f_i^+(u; \eta(E(x), u)) - \lambda_i g_i^+(u; \eta(E(x), u))) \\ & + \sum_{j=1}^m \beta_j h_j^+(u; \eta(E(x), u)) \geq 0, \quad \forall x \in X_0, \\ & f_i(u) - \lambda_i g_i(u) \geq 0, \quad i \in P, \\ & \beta_j h_j(u) \geq 0, \quad j \in M, \\ & \lambda = (\lambda_1, \dots, \lambda_p) \geq 0, \quad \alpha = (\alpha_1, \dots, \alpha_p) > 0, \\ & \beta = (\beta_1, \dots, \beta_m) \geq 0, \quad u \in X_0. \end{cases} \quad (5.1)$$

Denote the feasible set of problem (VFD) by X' .

Theorem 5.1. (Weak duality) Let $x \in X$, $(u, \alpha, \beta, \lambda) \in X'$, and $E(u) = u$. If $\sum_{i=1}^p \alpha_i (f_i - \lambda_i g_i)$ is a pseudo-semilocal E -preinvex function, and $\sum_{j=1}^m \beta_j h_j$ is a quasi-semilocal E -preinvex function, and they are all E - η -semidifferentiable at u , then $\frac{f(x)}{g(x)} \not\leq \lambda$.

Proof. We proceed by contradicting. Then, if the conclusion is not true, we have

$$\frac{f(x)}{g(x)} \leq \lambda, \quad \text{i.e.,} \quad f_i(x) - \lambda_i g_i(x) \leq 0, \quad \forall i \in P,$$

$$f_k(x) - \lambda_k g_k(x) < 0, \quad \text{for some } k \in P.$$

From $\alpha > 0$ and $(u, \alpha, \beta, \lambda) \in X'$, we get

$$\sum_{i=1}^p \alpha_i (f_i(x) - \lambda_i g_i(x)) < 0 \leq \sum_{i=1}^p \alpha_i (f_i(u) - \lambda_i g_i(u)).$$

By the pseudo-semilocal E -preinvexity of $\sum_{i=1}^p \alpha_i (f_i - \lambda_i g_i)$ and Lemma 3.1, we obtain

$$\left(\sum_{i=1}^p \alpha_i (f_i - \lambda_i g_i) \right)^+ (u; \eta(E(x), u)) < 0,$$

that is, $\sum_{i=1}^p \alpha_i (f_i^+(u; \eta(E(x), u)) - \lambda_i g_i^+(u; \eta(E(x), u))) < 0$.

On the other hand, from $\beta \geq 0$ and $x \in X$, it follows that

$$\sum_{j=1}^m \beta_j h_j(x) \leq 0 \leq \sum_{j=1}^m \beta_j h_j(u).$$

Using the quasi-semilocal E -preinvexity of $\sum_{j=1}^m \beta_j h_j$ and Lemma 3.1, one has

$$\left(\sum_{j=1}^m \beta_j h_j \right)^+ (u; \eta(E(x), u)) \leq 0, \quad \sum_{j=1}^m \beta_j h_j^+(u; \eta(E(x), u)) \leq 0.$$

Therefore,

$$\sum_{i=1}^p \alpha_i (f_i^+(u; \eta(E(x), u)) - \lambda_i g_i^+(u; \eta(E(x), u))) + \sum_{j=1}^m \beta_j h_j^+(u; \eta(E(x), u)) < 0,$$

which is in contradiction with $(u, \alpha, \beta, \lambda) \in X'$.

Similarly, we have the following weak duality.

Theorem 5.2. (Weak duality) Let $x \in X$, $(u, \alpha, \beta, \lambda) \in X'$, and $E(u) = u$. If $\sum_{i=1}^p \alpha_i (f_i - \lambda_i g_i) + \sum_{j=1}^m \beta_j h_j$ is pseudo-semilocal E -preinvex and E - η -semidifferentiable at u , then $\frac{f(x)}{g(x)} \not\leq \lambda$.

Theorem 5.3. (Converse duality) Let $\bar{x} \in X$ and $(x^*, \alpha^*, \beta^*, \lambda^*) \in X'$, $E(x^*) = x^*$, where $\lambda^* = \frac{f(x^*)}{g(x^*)} = \frac{f(\bar{x})}{g(\bar{x})} = (\lambda_1^*, \dots, \lambda_p^*)$. If $f_i - \lambda_i^* g_i$ ($i \in P$), h_j ($j \in M$) are all semilocal E -preinvex functions and all E - η -semidifferentiable at x^* , then \bar{x} is a weak efficient solution for (VFP).

Proof. By the given hypotheses and Lemma 3.1, for any $x \in X$, we get

$$\begin{aligned}(f_i(x) - \lambda_i^* g_i(x)) - (f_i(x^*) - \lambda_i^* g_i(x^*)) &\geq f_i^+(x^*; \eta(E(x), x^*)) - \lambda_i^* g_i^+(x^*; \eta(E(x), x^*)), \quad i \in P; \\ h_j(x) - h_j(x^*) &\geq h_j^+(x^*; \eta(E(x), x^*)), \quad j \in M.\end{aligned}$$

Utilizing the first constraint condition for (VFD), $\alpha^* > 0$, $\beta^* \geq 0$, $\lambda^* \geq 0$ and the two inequalities above, we have

$$\begin{aligned}\sum_{i=1}^p \alpha_i^* ((f_i(x) - \lambda_i^* g_i(x)) - (f_i(x^*) - \lambda_i^* g_i(x^*))) + \sum_{j=1}^m \beta_j^* (h_j(x) - h_j(x^*)) &\geq \\ \sum_{i=1}^p \alpha_i^* (f_i^+(x^*; \eta(E(x), x^*)) - \lambda_i^* g_i^+(x^*; \eta(E(x), x^*))) + \sum_{j=1}^m \beta_j^* h_j^+(x^*; \eta(E(x), x^*)) &\geq 0.\end{aligned}$$

In view of $h_j(x) \leq 0$, $\beta_j^* \geq 0$, $\beta_j^* h_j(x^*) \geq 0$ ($j \in M$) and $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)}$ ($i \in P$), one has

$$\sum_{i=1}^p \alpha_i^* (f_i(x) - \lambda_i^* g_i(x)) \geq 0, \quad \forall x \in X. \quad (5.2)$$

Now, we assume that \bar{x} is not a weak efficient solution for (VFP). From $\lambda_i^* = \frac{f_i(\bar{x})}{g_i(\bar{x})}$ ($i \in P$) and Lemma 4.1, it follows that \bar{x} is not a weak efficient solution for $(VFP)_{\lambda^*}$. Hence, there exists an $\tilde{x} \in X$ such that $f_i(\tilde{x}) - \lambda_i^* g_i(\tilde{x}) < f_i(\bar{x}) - \lambda_i^* g_i(\bar{x}) = 0$, $i \in P$, which implies that

$$\sum_{i=1}^p \alpha_i^* (f_i(\tilde{x}) - \lambda_i^* g_i(\tilde{x})) < 0.$$

This is a contradiction to the inequality (5.2).

So the theorem is proved.

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Authors' contributions

JHH proposed the concept of semilocal E-preinvex functions and discussed some of its important properties. Additionally, JHH carried out the studies of optimality and duality for a nonlinear multiobjective fractional programming with semilocal E-preinvex and related functions and drafted the manuscript. LSY participated in the design, coordination and examination of this paper. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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